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STRUCTURAL TRANSFORMATIONS OF DYNAMICAL SYSTEMS WITH GYROSCOPIC FORCES†

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A general method for the structural transformation of dynamical systems containing gyroscopic forces is considered. The method simplifies the investigation without changing the qualitative properties of the initial system. Examples are considered. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM. THE INITIAL MATRIX EQUATION

The method of averaging is one of the effective methods for the approximate investigation of problems of non-linear mechanics (in particular, mathematical models of various vibration processes. Bogolyubov showed that it can be applied to a special (standard) form of differential equations [1–4]

$$dx/dt = \mu X(t, x, \mu), \; x(0) = x_0 \tag{1.1}$$

Here $x \equiv \operatorname{colon}(x_1, \ldots, x_n)$, $X \equiv \operatorname{colon}(X_1, \ldots, X_n)$ are certain n-dimensional column vectors and μ is a small non-negative parameter. Averaging is carried out on the right-hand side of Eq. (1.1).

Preliminary reduction to the standard form (1.1) is a necessary stage when using the method of averaging. Moreover, a similar state sometimes accompanies stabilities (instabilities) of the average system in the initial system also, when the latter does not necessarily have the standard form (1.1). However, there are no general theorems which justify the use of the method of averaging in such cases, and there is not always a formal replacement of some equations by others which can lead to the correct result [5]. This applies, in particular, to dynamical systems with gyroscopic forces. If the coefficients of the gyroscopic terms in the equations of perturbed motion are periodic in t, with a certain real period τ, in the formal averaging over a period the gyroscopic terms may vanish, and consideration of these despite their periodicity, nevertheless leads to some stabilizing effect [6, 7].

When there are gyroscopic terms in the initial equations it is desirable to have available fundamental methods which would enable one, without changing the stability conditions and the stabilizing properties, inherent in gyroscopic structures, to change the initial equations so that the transformed equations do not contain gyroscopic terms at all. In this report it is worth noting Bulgakov's method of normal coordinates, which can be extended to systems with gyroscopic terms, and by means of which one can obtain equations which are easily reducible to the standard form (1.1) [8]. With certain reservations, Bulgakov's method can be extended to systems with slowly varying parameters [9], although, strictly speaking, it is only applicable to systems of differential equations with constant coefficients.

We note also the method of reducing gyroscopic systems to standard form proposed in [4] when the matrix of the gyroscopic forces is non-degenerate and constant.

Below we consider, under more general assumptions, a method for the structural transformation of the initial equations which leads to the elimination from them of the gyroscopic terms and also of the non-conservative positional terms. The object of the investigation is a matrix equation of the form

$$a_0\ddot{x} + D\dot{x} + H\dot{x} + \Pi x + Px = F(t) + X(x, \dot{x}) \tag{1.2}$$

where x is an n-dimensional vector, a_0 is a certain positive scalar parameter, D and Π are $n \times n$ symmetric matrices, H and P are skew-symmetric matrices of the same dimensions, $F(t)$ is an n-dimensional vector of the perturbing forces and $X(x, \dot{x})$ is a vector function containing x and $\dot{x}$ in powers higher than the first. The constancy of the matrices D, H, Π and P is not obligatory: their elements can be real, continuous and bounded functions of t. Equation (1.2) describes the motion of many mass systems, acted upon by

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dissipative, gyroscopic, potential and non-conservative positional forces, and also specified perturbing forces.

2. ORTHOGONAL TRANSFORMATION OF EQ. (1.2)

In Eq. (1.2) we will change to a new variable ξ by means of the transformation

$$\xi = Lx \tag{2.1}$$

where the matrix L will be defined later. As a result, we obtain the equation

$$\begin{aligned} & a_0\ddot{\xi} + LDL^{-1}\dot{\xi} + (L\Pi - a_0\ddot{L})L^{-1}\xi + (LH - 2a_0\dot{L})L^{-1}(\dot{\xi} - \dot{L}L^{-1}\xi) + \\ & + L(P - DL^{-1}\dot{L})L^{-1}\xi = LF + \Xi \end{aligned} \tag{2.2}$$

in which the vector function Ξ contains the quantities ξ and $\dot{\xi}$ in powers no lower than the second.

Referring to Eq. (2.2), we note that the fourth term on the left-hand side in any case vanishes if the following condition is satisfied

$$\dot{L} = (2a_0)^{-1} LH \tag{2.3}$$

which, for a specified matrix H, can be regarded as a matrix equation in L. Taking into account the fact that H is skew-symmetric and satisfying, in the solutions of Eq. (2.3), the identity matrix E of the initial conditions, we obtain L in the form of an orthogonal matrix. In fact, if L is orthogonal, then by its definition we must have $LL^T = E$, where L^T is the transposed matrix of L. Differentiating this expression and taking into account the fact that (2.3) yields $\dot{L}^T = (2a_0)^{-1}H^TL^T$, we obtain the following expression which vanishes identically

$$(2a_0)^{-1}(LHL^T + LH^TL^T) = 0$$

since, by virtue of the fact that the matrix of the gyroscopic forces is skew-symmetric we have $H = -H^T$. Hence $LL^T = C$, where C is a certain constant matrix. Satisfying here the identity matrix of the initial conditions, we have $LL^T = E$, which was required. Since $|\det L| = 1$, when $L(t)$ and $\dot{L}(t)$ are bounded in the interval (t_0, ∞), the matrix L will be the Lyapunov matrix.

Taking condition (2.3) into account we can write the matrix equation (2.2) in the form

$$a_0\ddot{\xi} + LDL^T\dot{\xi} + \{L(\Pi + P - DL^T\dot{L}) - a_0\ddot{L}\}L^T\xi = LF + \Xi \tag{2.4}$$

If there are no dissipative, non-conservative forces and also no external perturbation, Eq. (2.4) can be written in the form

$$\ddot{\xi} + K\xi = \Xi, \quad K = a_0^{-1}(L\Pi - a_0\ddot{L})L^T \tag{2.5}$$

In the case of the symmetric matrix K, Eq. (2.5), apart from the right-hand side, corresponds to a linear Hamiltonian equation.

Note that when

$$2a_0\dot{L} - LH = 0, \quad DL^{-1}\dot{L} - P = 0 \tag{2.6}$$

the last two terms on the left-hand side of Eq. (2.2) vanish and, hence, not only are the gyroscopic terms eliminated from it but also the non-conservative positional terms. However, conditions (2.6) are only satisfied simultaneously in the exceptional case when the matrices H, D and P, without being identically zero, are related by the equation

$$2a_0P - DH = 0 \tag{2.7}$$

3. THE LAGRANGE GYROSCOPE ON A VIBRATING BASE

The procedure described in Sections 1 and 2 can be applied to the problem of stabilizing a Lagrange gyroscope, assuming it to be on a base undergoing a vertical harmonic vibration as given by $\zeta = a\cos pt$, where a and p are the amplitude and angular frequency of the vibrations, respectively. Ignoring the resistance of the medium, the first approximation of the equations of the perturbed motion, obtained from the Euler–Poisson system, has the following form for the case in question [10]

$$\ddot{\gamma} - \frac{2A-C}{A}\omega\dot{\gamma}_2 + \frac{1}{A}\{(C-A)\omega^2 - m(g - ap^2\cos pt)z_c\}\gamma_1 = 0$$
$$\ddot{\gamma}_2 + \frac{2A-C}{A}\omega\dot{\gamma}_1 + \frac{1}{A}\{(C-A)\omega^2 - m(g - ap^2\cos pt)z_c\}\gamma_2 = 0 \tag{3.1}$$

Here A, A and C are the principal moments of inertia of the gyroscope about the orthogonal axes Ox, Oy and Oz, connected with it, with origin at the fixed point O, m is the mass of the gyroscope, γ_1 and γ_2 are the direction cosines of the ascending vertical $O\zeta$ with the Ox and Oy axes, and z_c is the coordinate of the centre of gravity of the body about the Oz axis (we will henceforth assume $z_c = l > 0$). Since the resistance to motion is ignored in the calculation, we can assume the angular velocity ω of the gyroscope to be constant. The general algorithm, which leads, in the special case of Eqs (3.1), to the elimination of the gyroscopic terms from them, is described by matrix equation (2.3). Assuming $L = \| l_{jk} \|_1^2$ and also

$$a_0 = 1,\quad H = \begin{Vmatrix} 0 & -h \\ h & 0 \end{Vmatrix},\quad h = \frac{2A-C}{A}\omega \tag{3.2}$$

we arrive at the following equations in l_{jk}

$$2\dot{l}_{11} = hl_{12},\quad 2\dot{l}_{12} = -hl_{11},\quad 2\dot{l}_{21} = hl_{22},\quad 2\dot{l}_{22} = -hl_{21} \tag{3.3}$$

Equations (3.3) can be rapidly integrated. By satisfying the identity matrix of the initial conditions with respect to L, we obtain

$$L = \begin{Vmatrix} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{Vmatrix},\quad \Omega = \frac{h}{2} = \frac{2A-C}{2A}\omega \tag{3.4}$$

Apart from higher-order terms, Eq. (2.5) for the case considered can be represented in the form

$$\ddot{\xi} + (cE - \ddot{L}L^T)\xi = 0,\quad c = \frac{1}{A}\{(C-A)\omega^2 - m(g - ap^2\cos pt)l\}$$

where $\xi = [\xi_1, \xi_2]^T$ is a two-dimensional vector, E, as before, is the identity matrix, and L is defined by (3.4). As a result, putting $pt = 2z - \pi$, we obtain two scalar Mathieu equations of the same structure

$$\frac{d^2\xi_k}{dz^2} + (\nu - 2q\cos 2z)\xi_k = 0,\quad k = 1,2 \tag{3.5}$$

$$\nu = \frac{1}{A^2p^2}(C^2\omega^2 - 4Amgl),\quad q = \frac{2mal}{A} \tag{3.6}$$

We will further assume that the following condition is satisfied

$$C^2\omega^2 - 4Amgl < 0 \tag{3.7}$$

which, when there are no vibrations, corresponds to instability in the motion of the Lagrange gyroscope [11, 12].

We will show that, using the vertical vibration, we can stabilize the motion for this case. We will use for this purpose an Ince–Strutt diagram, as it applies to Eqs (3.5) [11, 13, 14].

Figure 1 shows part of this diagram, enclosing the region of negative values of ν. For these values of ν and small values of the parameter q, the part of the diagram shown hatched in the figure corresponds to the region of stability and is bounded by the parabola $q^2 = -2\nu$ and the straight line $q = 1 - \nu$. Stability (non-asymptotic) will be guaranteed if the point $M(\nu, q)$ lies within the hatched part of the diagram. For $\nu < 0$ this will always occur if the point in question is situated above the parabola $q^2 = -2\nu$ and below the straight line $q = 1 - \nu$, which leads to the condition

$$1-\nu > \frac{2mal}{A} > \sqrt{-2\nu} \tag{3.8}$$

Taking into account the notation in (3.6) we conclude that when $mal < A$ the left-hand side of inequality (3.8) is always satisfied if condition (3.7) is satisfied; the right-hand side leads to a lower limit of the vibration frequency [10]

$$p > \frac{1}{2mal}\sqrt{2(4Amgl - C^2\omega^2)} \tag{3.9}$$

where, by assumption, the expression under the square root sign must be assumed to be positive. In this case, rotation of the body plays a useful role since, as ω increases, the value of the angular frequency of vibration required to stabilize the motion decreases.

Some special cases can be derived from (3.9). For example, if the gyroscope is not, in general, rotated, which corresponds to $\omega = 0$, then assuming $A = m\rho^2$, where ρ is the equatorial radius of inertia of the body, we arrive at a condition which is identical with the well-known condition for the stability of the vertical position of a physical pendulum, set up on a base which is subject to vertical vibrations [15]

$$ap > \sqrt{2g\rho^2/l}$$

If we put $\rho = l$ here, we obtain the Bogolyubov–Kapitsa condition for the case of a mathematical pendulum [16].

4. THE TRANSFORMED EQUATIONS OF A POINT MASS IN A UNIFORMLY ROTATING SYSTEM OF COORDINATES

We will assume that a point mass m undergoes motion relative to a system of coordinates $Ox_1x_2x_3$, rotating with constant angular velocity ω with origin at an arbitrary point in space, due to the action of a specified force $F = F(t)$. The equations of relative motion of the point mass in projections onto the axis of a moving trihedron $Ox_1x_2x_3$ have the form

$$m\ddot{x}_1 - \partial T_0/\partial x_1 - 2m\omega_3\dot{x}_2 + 2m\omega_2\dot{x}_3 = F_1 \quad (1\ 2\ 3) \tag{4.1}$$

$$T_0 = \tfrac{1}{2}m\{(\omega_2 x_3 - \omega_3 x_2)^2 + (\omega_3 x_1 - \omega_1 x_3)^2 + (\omega_1 x_2 - \omega_2 x_1)^2\}$$

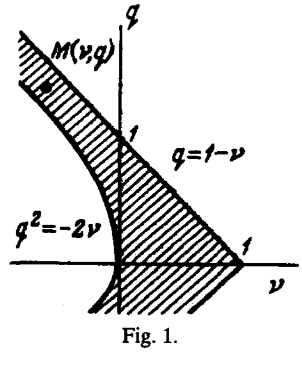

Fig. 1.

where F_1, F_2 and F_3 are the projections of the specified force onto the axes; the derivatives $\partial T_0/\partial x_j$ are the projections of the centrifugal force, developed due to the rotation of the trihedron $Ox_1x_2x_3$ onto the corresponding axes.

Equations in the form (4.1) turn out to be inconvenient for constructing exact solutions. Hence, when considering different kinds of problems, described by system (4.1) (for example, the motion of a heavy point in the regions of the Earth's surface taking its rotation into account), the method of iterations is preferable to direct integration of the system. The method described in Section 2 enables us, by changing Eqs (4.1), to simplify their investigation considerably.

Comparing system (4.1) with Eq. (1.2), we put

$$H = 2m\begin{Vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Vmatrix} \tag{4.2}$$

From Eq. (2.3), by means of which the gyroscopic terms in matrix equation (1.2) are eliminated, we obtain, taking (4.2) into account, three groups of similar equations defining the elements of the orthogonal matrix $L = \| l_{jk} \|_1^3$ for the specified problem

$$\dot{l}_{11} = \omega_3 l_{12} - \omega_2 l_{13}, \quad \dot{l}_{12} = -\omega_3 l_{11} + \omega_1 l_{13}, \quad \dot{l}_{13} = \omega_2 l_{11} - \omega_1 l_{12} \quad (1\ 2\ 3)$$

Hence we have a single characteristic equation for these groups

$$\lambda(\lambda^2 + \omega^2) = 0, \quad \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} \tag{4.3}$$

Equation (4.3) has a simple zero root $\lambda_1 = 0$ and two pure imaginary roots $\lambda_{2,3} = \pm i\omega$. By satisfying the identity matrix of the initial conditions we obtain

$$L = \frac{1}{\omega^2}\begin{Vmatrix} s_1 & n_{12} - m_3 & n_{13} + m_2 \\ n_{12} + m_3 & s_2 & n_{23} - m_1 \\ n_{13} - m_2 & n_{23} + m_1 & s_3 \end{Vmatrix} \tag{4.4}$$

Here, for brevity, we have put

$$s_j = \omega_j^2 + (\omega^2 - \omega_j^2)\cos\omega t, \quad m_j = \omega\omega_j \sin\omega t, \quad n_{jk} = \omega_j\omega_k(1 - \cos\omega t), \quad j = \overline{1,3} \tag{4.5}$$

We further have

$$\Pi = m\begin{Vmatrix} -(\omega_3^2 + \omega_2^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_2\omega_1 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_3\omega_1 & \omega_3\omega_2 & -(\omega_2^2 + \omega_1^2) \end{Vmatrix} \tag{4.6}$$

Equation (2.4), apart from the non-linear vector on the right-hand side, can be represented in the form

$$m\ddot{\xi} + (L\Pi - m\ddot{L})L^T\xi = LF, \quad \xi = [\xi_1, \xi_2, \xi_3]^T, \quad F = [F_1, F_2, F_3]^T \tag{4.7}$$

By operating with matrices (4.4) and (4.6) it can be shown that the expression in parentheses in Eq. (4.7) vanishes, as a result of which this equation takes the simple form

$$m\ddot{\xi} = LF \tag{4.8}$$

corresponding to three scalar equations, that are modified compared with system (4.1). Since the acting force, and also the angular velocity of the trihedron $Ox_1x_2x_3$, are assumed to be specified, the exact solution of matrix equation (4.8) is given by the binary quadrature

$$\xi = \xi(0) + \dot{\xi}(0)t + \frac{1}{m}\int_0^t\int_0^t LF\,dt^2 \tag{4.9}$$

where $\xi(0)$ and $\dot{\xi}(0)$ are the values of the vectors ξ and $\dot{\xi}$ at the initial instant. Reverting to the initial matrix variable we have, by virtue of (2.1)

$$x = L^{-1}\xi = L^T\xi$$

For illustration we will consider a special case of Eqs (4.1) relating to the motion of a heavy point in the neighbourhood of the Earth, taking its rotation into account [18]. As it applies to this case, the origin of the trihedron $Ox_1x_2x_3$ is chosen to be at a certain arbitrary point in the region of the Earth's surface, which approximates the figure of the Earth by a sphere. We direct the Ox_1 axis in the meridian plane towards the north, the Ox_2 towards the east, and the Ox_3 axis along the geocentric vertical to the centre of the Earth's sphere. We will assume

$$\omega_1 = \omega\cos\varphi, \ \omega_2 = 0, \ \omega_3 = -\omega\sin\varphi$$

where ω is the angular velocity of rotation of the Earth and φ is the geocentric latitude. Further, we must carry out other quadratures indicated in (4.9), assuming $F = (0, 0, mg)$, where g is the gravitational acceleration, and then revert to the initial variable.

For simplicity we will assume that when $t = 0$

$$x_1 = x_2 = x_3 = 0, \ \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0 \tag{4.10}$$

This corresponds to the heavy point being situated at the origin of the trihedron $Ox_1x_2x_3$. According to representation (2.1) the same initial conditions also correspond to the components of the vector ξ. Further we must carry out simple calculations, guided by expression (4.9) and taking the initial conditions (4.10) into account. As a result we obtain the exact formulae

$$\begin{aligned} x_1 &= \frac{g}{2\omega^2}\left\{(1-\cos\omega t)\cos\omega t + (\omega t - \sin\omega t)\sin\omega t - \frac{\omega^2 t^2}{2}\right\}\sin 2\varphi \\ x_2 &= \frac{g}{\omega^2}(\sin\omega t - \omega t\cos\omega t)\cos\varphi \\ x_3 &= \frac{g}{\omega^2}\left\{(\cos\omega t - 1 + \omega t\sin\omega t)\cos^2\varphi + \frac{\omega^2 t^2}{2}\sin^2\varphi\right\} \end{aligned} \tag{4.11}$$

in which we must, of course, assume $\omega \neq 0$.

Assuming small values of ωt ($\omega \approx 7.29 \times 10^5 \ \text{s}^{-1}$) we can obtain the well-known approximate expressions for x_1, x_2 and x_3 from formulae (4.11).

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